Strictly Real Locally Convex Algebras

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Abstract. Different kinds of strictly real locally convex algebras are considered. We extend, to this more general context, the basic result of I. Kaplansky, in the Banach case, that is the commutativity modulo the Jacobson radical.

Key words: Strictly real, algebras, m-convex, A-convex, uniformly convex.

1. Introduction

In 1949, I. Kaplansky has introduced [10] the class of strictly real Banach algebras. Since then, not much work has been done on them. We intend, in this paper, to examine locally convex algebras which are not Banach, but still have the same or similar properties. These are often encountered in different branches of functional analysis, especially in spectral theory.

In Section 3, we show that the (real) multiplier algebra $M(E)$, of a semisimple strictly real Banach algebra, is also strictly real (Proposition 10). This algebra $M(E)$ will be endowed with another topology in Section 6, and serve as an example of a different kind.

Section 4 is devoted to $m$-convex algebras. We obtain Kaplansky’s result (as in the Banach case), that is the commutativity modulo the (Jacobson) radical of any complete strictly real $l.m.c.a.$ (Proposition 13). There is also an observation on Michael’s problem (Proposition 16). The same results are obtained for locally $A$-convex algebras, in Section 5 (Proposition 22, Proposition 24).

Locally uniformly $A$-convex algebras are examined in Section 6. In that context, Mackey completeness is sufficient to get Kaplansky’s result (Proposition 27).

In Section 7, we consider locally uniformly convex algebras introduced in [19] (see also [21]). The same results are obtained (Proposition 40), as for locally uniformly $A$-convex algebras. We also provide an example of a such sequentially complete algebra, which has not been done in [19] and [21].

2. Preliminaries

Let $(E, \tau)$ be a locally convex algebra (l.c.a.), with separately continuous multiplication, whose topology $\tau$ is given by a family $(p_\lambda)_{\lambda \in \Lambda}$ of seminorms. For simplicity and provided there is nowhere
a risk of confusion, we will write only \((p_\lambda)_\lambda\) instead of \((p_\lambda)_{\lambda \in \Lambda}\). The algebra \((E, \tau)\) is said to be locally \(A\)-convex (l-\(A\)-c.a.; [4], [5]) if, for every \(x\) and every \(\lambda\), there is \(M(x, \lambda) > 0\) such that

\[
\max [p_\lambda(xy), p_\lambda(yx)] \leq M(x, \lambda)p_\lambda(y); \forall y \in E.
\]

In the case of a single vector space norm, \((E, \|\cdot\|)\) is called an \(A\)-normed algebra. If \(M(x, \lambda) = M(x)\) depends only on \(x\), we say that \((E, \tau)\) is a locally uniformly \(A\)-convex algebra (l.u-A-c.a.; [5]). If it happens that, for every \(\lambda\),

\[
p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y); \forall x, y \in E.
\]

then \((E, \tau)\) is named a locally \(m\)-convex algebra (l.m.c.a.; cf. [12], [11]). Recall also that a l.c.a. has continuous multiplication if, for every \(\lambda\), there is \(\lambda'\) such that

\[
p_\lambda(xy) \leq p_{\lambda'}(x)p_{\lambda'}(y); \forall x, y \in E.
\]

A l.c.a. is said to be locally uniformly convex (l.u.c.a.; [19], [21]) if \((\forall x)(\exists M(x) > 0)(\forall \lambda)(\exists \lambda') : p_\lambda(xy) \leq M(x)p_{\lambda'}(y); \forall y \in E\), with \(\lambda'\) depending only on \(\lambda\) but not on \(x\).

An element \(x\) of \(E\) is said to be bounded [1] (\(i\)-bounded in the sense of S. Warner [24]) if there is \(\alpha > 0\) such that \(\{(\alpha x)^n : n = 1, 2, \ldots\}\) is bounded.

The bounded structure (bornology) of a locally convex space (l.c.s.) \((E, \tau)\) is the collection \(\mathbb{B}_\tau\) of all the subsets \(B\) of \(E\) which are bounded in the sense of Kolmogorov-von Neumann, that is \(B\) is absorbed by every neighborhood of the origin. If \(\tau_{\|\cdot\|}\) is the topology induced by a norm \(\|\cdot\|\), we write \(\mathbb{B}_{\|\cdot\|}\). We say that a l.c.s. \((E, \tau)\) is Mackey complete (\(M\)-complete) if its bounded structure \(\mathbb{B}_\tau\) admits a fundamental system \(\mathcal{B}\) of Banach discs that is, for every \(B\) in \(\mathcal{B}\), the vector space generated by \(B\) is a Banach space when endowed with the gauge \(\|\cdot\|_B\) of \(B\). For all notions concerning bornology, see [8].

As in the Banach case, a l.c.a. \(E\) is said to be strictly real if \(Sp(x) \subset \mathbb{R}\), for every \(x \in E\). Actually, the notion of strict reality can be defined without any reference to a given topology. It is essentially algebraic. A real algebra is said to be strictly real if \(Sp(x) \subset \mathbb{R}\), for every \(x \in E\), where \(Sp(x)\) is the spectrum of \(x \in E\) in the complexification \(E_\mathbb{C}\) of \(E\). The fundamental property is the non voidness of \(Sp_\mathbb{R}(x) = \{\alpha \in \mathbb{R} : x - \alpha\) is not invertible\}. Here are some general facts, already noticed in the Banach case.

**Lemma 1.** Let \(E\) be a real algebra such that \(Sp_\mathbb{R}(x) \neq \emptyset\), for every \(x \in E\). Then the following are equivalent

(i) \(E\) is strictly real.

(ii) \(Sp(x^2) \subset \mathbb{R}_+, \) for every \(x \in E\).

(iii) \(e + x^2\) is invertible, for every \(x \in E\).
We give here some examples. Others will be given in different contexts.

**Example 2.** Let \( E = C_b(\mathbb{R}, \mathbb{R}) \) be the algebra of continuous bounded real functions on the real line \( \mathbb{R} \). Endow it with the topology given by the family of seminorms \( (p_\Phi)_\Phi \), \( \Phi \in C^0_b(\mathbb{R}, \mathbb{R}) \), where \( C^0_b(\mathbb{R}, \mathbb{R}) \) is the algebra of functions \( \Phi \in E \) vanishing at infinity and
\[
p_\Phi(f) = \sup\{|\Phi(x)| | f(x) | : x \in \mathbb{R} \}.
\]
It is a \( l.u.-A.c.a. \) which is not a \( l.m.c.a. \)[4].

**Example 3.** Let \( E = C(\mathbb{R}, \mathbb{R}) \) be the algebra of continuous real valued functions on the real line \( \mathbb{R} \). Endow it with the topology of uniform convergence on compacta. This is a \( l.m.c.a. \) which is not a \( l.u.-A.c.a. \).

**Example 4.** Considering the standard cartesian product \( C_b(\mathbb{R}, \mathbb{R}) \times C(\mathbb{R}, \mathbb{R}) \), one obtains a \( l-A.c.a \) which is not \( m \)-convex nor uniformly \( A \)-convex.

**Example 5.** The following example, due to S. Warner [21] in the context of \( l.m.c.a. \)'s, is a strictly real \( l.c.a. \) which is not locally \( A \)-convex. Let \( E = C[0, 1] \) be the Banach algebra of continuous real functions on the interval \([0, 1]\), endowed with the uniform topology. If \( E' \) is the topological dual of \( E \) and \( \sigma = \sigma(E, E') \) is the associated weak topology, then \((E, \sigma)\) is not a \( l.m.c.a. \) ([23], p. 1028). Now, by a result of S. Warner ([23], Theorem 1) and another one of A. C. Cochran ([6], Theorem 3.4), it can not be a \( l-A.c.a. \).

**Remark 6.** To enlarge the class of strictly real \( l.c.a. \)'s, one notices that the property is kept by the following operations: Adjunction of unity, cartesian product, projective limit, the image by an algebra morphism, ...

3. **Banach algebras**

This section is devoted to the (real) multiplier algebra \( M(E) \), of a commutative strictly real Banach algebra \( E \) without order. In the complex case, a notable use is made of the carrier space \( \mathcal{M}(E) \). In the real case, the latter can be void. But if we depart with a real algebra such that \( \mathcal{M}(E) \) is separating, then there is no obstacle to reproduce the arguments in [22]. Indeed, they are of an algebraic or a general topological character. No appeal is made to complex analysis. So, one can show here that the (real) multiplier algebra of \( M(E) \), of \( E \), is also strictly real. But first a concrete example. Recall that an algebra \( E \) is said to be without order if \( x = 0 \) whenever \( xy = 0 \) for every \( y \). A linear map \( S \) from \( E \) to \( E \) is named a multiplier if \( S(xy) = xS(y) \), for every \( x \) and \( y \) in \( E \). All multipliers are automatically continuous, and the algebra \( M(E) \) is commutative [22].

**Example 7.** Let \( E = C_0(\mathbb{R}, \mathbb{R}) \) be the algebra of continuous real valued functions on the real line \( \mathbb{R} \), vanishing at infinity. Endowed with the topology of the supremum norm \( ||.||_\infty \), it becomes a Banach algebra. The evaluations \( \chi_x \), on \( E \), given by \( \chi_x(f) = f(x) \), are characters of \( E \). Hence \( \mathcal{M}(E) \) is separating. Actually, \( E \) is a semisimple strictly real Banach algebra \( E \) without order.
Remark 8. Though not used in this paper, we give the following simple but interesting facts.

1. The kernel $\ker T$, of a multiplier $T$ of $E$, is a bilateral ideal. This is due to $T(xy) = T(x)y$, for every $x$ and every $y$.

2. If $(E, \| \cdot \|)$ is a strictly real Banach algebra and $T$ is a multiplier $E$, then $T(\text{Rad}(E)) \subset \text{Rad}(E)$. Indeed, take $x \in \text{Rad}(E)$. One has to show that $\varrho(T(x)y) = 0$, for every $y$. But $\varrho(T(x)y) = \varrho(xT(y))$. Now $E$ being strictly real (hence commutative modulo its radical), one has $\varrho(xT(y)) \leq \varrho(x)\varrho(T(y))$, with $\varrho(x) = 0$.

Remark 9. We now give a simple property, we will use in the sequel. It is general and has an interest in its own. If $E$ is a real Banach algebra, then $E/\text{Rad}(E)$ is without order. Indeed if $\bar{x} \in E/\text{Rad}(E)$ is such that $\bar{x}\bar{y} = \bar{y}\bar{x} = 0$ for every $\bar{y}$, then $xy \in \text{Rad}(E)$, for every $y$. Whence $x \in \text{Rad}(E)$. So $\bar{x} = \bar{0}$.

Proposition 10. Let $(E, \| \cdot \|)$ be a strictly real Banach algebra. Then

(i) The quotient algebra $M(E)/\text{Rad}(E)$ is also strictly real.

(ii) If moreover $E$ is semisimple, then $M(E)$ is strictly real.

Proof. (i) The algebra $F = E/\text{Rad}(E)$ is commutative [10]. So it is a semisimple commutative Banach algebra. It is also without order (Proposition 9). Now, exactly as in [18], every multiplier $T \in M(F)$ can be identified with a continuous bounded real valued function on the carrier space $\mathcal{M}(F)$ of $F$. So $\text{Sp}T = \text{Sp}g \subset \mathbb{R}$.

(ii) Immediate from (i).

Remark 11. By a general and simple construction, we can obtain non commutative strictly real Banach algebras from commutative ones. Let $E$ be a strictly real commutative Banach algebra that admits a unital isometric morphism $T$ (i.e., a morphism such that $\|T(x)\| = \|x\|$, for every $x$) which is not a multiplier. Endow $E$ with the multiplication $\times_T$ given by $x \times_T y = xT(y)$. It is not commutative. One easily checks that $E$ remains strictly real.

We do not know, in general, how strict reality behaves concerning completion. We can say the following.

Proposition 12. Let $(E, \| \cdot \|)$ be a strictly real normed algebra. If $E$ is commutative, then the completion $\hat{E}$ of $E$ is also strictly real.

Proof. Let $x \in \hat{E}$, with $x = \lim x_n$, $x_n \in E$. One has $\text{Sp}(x) = \{ \chi(x) : \chi \in \mathcal{M}(\hat{E}) \}$ and $\chi(x) = \lim \chi(x_n)$. Since, by hypothesis, every $\chi(x_n) \in \mathbb{R}$, one has $\chi(x)$ for every $\chi \in \mathcal{M}(\hat{E})$. 

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4. **Locally m-convex algebras**

First, let us recall the Arens-Michael decomposition [12] of a complete l.m.c.a. $(E, (\| \cdot \|_\lambda))$. Put $N_\lambda = \{ x \in E : \| x \|_\lambda = 0 \}$ and $E_\lambda = E/N_\lambda$ endowed with the algebra norm $\| . \|_\lambda$ given by $\| x + N_\lambda \|_\lambda = \| x \|_\lambda$. The norm in the completion $E_\lambda^\wedge$ of $E_\lambda$ is still denoted $\| . \|_\lambda$. Then

$$(E, (\| \cdot \|_\lambda)) = \lim_i (E_i, \| . \|_i).$$

Denote by $\pi_\lambda$ the canonical surjection $x \mapsto \bar{x} = x + N_\lambda$, from $E$ to $E_\lambda^\wedge$.

**Proposition 13.** Let $(E, (\| \cdot \|_\lambda))$ be a real complete l.m.c.a.. Then

(i) $E$ is strictly real if, and only if, every factor $E_\lambda^\wedge$ is a strictly real Banach algebra.

(ii) If $E$ is strictly real, then it is commutative modulo its (Jacobson) radical $Rad(E)$.

**Proof.**

(i) If $x = (x_\lambda)_\lambda \in E$, then $Sp(x) = \cup_\lambda Sp(x_\lambda)$. Whence the claim.

(ii) If $x = (x_\lambda)_\lambda \in E$ and $y = (y_\lambda)_\lambda \in E$, then $xy - yx = (x_\lambda y_\lambda - y_\lambda x_\lambda)_\lambda$. Now, since $E_\lambda^\wedge$ is a strictly real Banach algebra, one has $x_\lambda y_\lambda - y_\lambda x_\lambda \in Rad(E_\lambda^\wedge)$, by a result of I. Kaplansky [6].

But ([8], Proposition 7.3, p. 29),

$$Rad(E) = \cap_\lambda \pi_\lambda^{-1}(Rad(E_\lambda^\wedge)).$$

Whence $xy - yx \in Rad(E)$.

We have an application to topological inductive limits of Banach algebras. But first, a general result. Recall that $(E, \tau) = \lim_i (E_i, \| . \|_i)$ is a topological strict inductive limit of Banach algebras if the restriction of the topology $\tau$ to each $E_i$ is exactly $\tau_{\| . \|_i}$ given by the norm $\| . \|_i$.

**Proposition 14.** Let $(E, \tau) = \lim_i (E_i, \| . \|_i)$ be a topological strict inductive limit of Banach algebras. If the set $I$ of indices is well ordered, then $E$ is strictly real.

**Proposition 15.** Let $(E, \tau) = \lim_n (E_n, \| . \|_n)$ be a topological strict inductive limit of a sequence of strictly real Banach algebras. Then $E$ is commutative modulo its (Jacobson) radical $Rad(E)$.

**Proof.** By the previous proposition, it is strictly real. By a result of Arosio [3] it is a l.m.c.a.. Then use (ii) of Proposition 13.

Another consequence is in relation with Michael’s problem [12]. Using Do Sin-Sya’s technique, T. Husain has shown [9] that every (real) character of a sequentially complete l.m.c.a. is bounded, when such character exists. Now Proposition 13 ensures that existence in our context. So, it is worthwhile to state the following. Observe also that the complex case is included.

**Proposition 16.** Let $(E, \tau)$ be a complete strictly real l.m.c.a.. Then
(i) Every (real) character of $E$ is bounded.

(ii) Every (real) character of the complexification $E_C$, of $E$, is bounded.

Proof. (i) cf. [9] or [13].

(ii) Actually, there is a bijection between $\mathcal{M}(E)$ and $\mathcal{M}(E_C)$.

Example 17. $C([0,1],\mathbb{R})$ endowed with the topology of uniform convergence on denumerable compact subsets of $[0,1]$ is a complete l.m.c.a.. Actually it is a locally $C^*$-algebra. So it can not be a $Q$-algebra for it should then be normable.

Example 18. Let $\Omega$ be the first uncountable ordinal and endow $[0,\Omega]$ with the order topology. Then the topological algebra $(C[0,\Omega],\mathbb{R})$, endowed with the compact-open topology, is a unital commutative and complete l.m.c.a.

To let Proposition 13 be meaningful, one has to exhibit non commutative strictly real l.m.c.a.’s.

Example 19. Take $F$ the real Banach algebra of $n \times n$ upper triangular matrices $M = (a_{ij})$ such that $a_{ij} \in \mathbb{C}$ for $i \neq j$ and $a_{ii} \in \mathbb{R}$, endowed with the norm $\|M\|$ given by $\|M\| = \max_i \Sigma_j |a_{ij}|$. Consider the standard cartesian product $E = \Pi_n F_n$, where $n \in \mathbb{N}^*$ and $F_n = F$ for every $n$. Then $(E,\tau)$, $\tau$ being the product topology, is a non commutative strictly real Fréchet l.m.c.a.

Example 20. Take the algebra $F$ in the previous example. Put $E_n = F \times \ldots \times F$ ($n$ times) and $E = \lim(E_n,(f_{nm})_{n,m})$, where $f_{nm}: E_n \rightarrow E_m$ are the standard projections, for $m \leq n$. Then $(E,\tau)$, $\tau$ being the projective topology, is a non commutative strictly real Fréchet l.m.c.a.

Remark 21. To obtain more examples, one can use Remark 3.5, stability properties of the $m$-convex structure and Proposition 13.

5. Locally $A$-convex algebras

These algebras have been introduced by A.C. Cochran, R. Keown and C.R. Williams [4]. Their study is actually reduced to $m$-convex ones [10] (see also [18]). As in the complex case, there is an $m$-convex topology $M(\tau)$ stronger than $\tau$. If the latter is given by a family $(p_\lambda)$ of seminorms (cf. Preliminaries), one may suppose that $p_\lambda(e) = 1$ for every $\lambda$, where $e$ is the unit element. Then one has $p_\lambda(x) \leq M(\lambda,x)$, for every $\lambda$. Put $q_\lambda(x) = \sup\{ p_\lambda(xu) : p_\lambda(u) \leq 1 \}$. The topology $M(\tau)$ is given by the family $(q_\lambda)$ of submultiplicative seminorms. Moreover, it is complete if $\tau$ is. So, using (ii) of Proposition 13, one obtains the following.

Proposition 22. A strictly real complete $l$-$A$-c.a. $(E,(|.|_\lambda)_\lambda)$ is commutative modulo its (Jacobson) radical $\text{Rad}(E)$. 
Remark 23. Recall that an $A$-normed algebra is complete if and only if it is a (Banach) normed algebra. It is known that a non-$m$-convex $l$-$A$-$c:a.$ $(E, (|.|)_A)$ is isomorphic to a subalgebra of the cartesian product of the $A$-normed algebra’s $E_\lambda = E/N_\lambda$, where $N_\lambda = \{ x \in E : |x|_\lambda = 0 \}$. But the $E_\lambda$’s are not complete, even if $(E, (|.|)_A)$ is, for otherwise the latter should be $m$-convex. So one can not apply Kaplansky’s result. It is the existence of the stronger $m$-convex topology $M(\tau)$ that allows it.

Concerning Michael’s problem, one has also a positive answer.

Proposition 24. Let $(E, \tau)$ be a complete strictly real $l$-$A$-$c:a.$ Then

(i) Every (real) character of $E$ is bounded.

(ii) Every (real) character of the complexification $E_\mathbb{C}$, of $E$, is bounded.

Proof. (i) Since $(E, \tau)$ is complete, the topologies $\tau$ and $M(\tau)$ have the same bounded sets [20]. Then use Proposition 16.

(ii) By (ii) of Proposition 16. \qed

Example 25. Let $E$ be a strictly real $l.u$-$A$-$c:a.$ that is not a $l.m.c.a$ and $F$ a strictly real $l.m.c.a$ which is not uniformly $A$-convex. Then the standard cartesian product algebra $E \times F$ is a strictly real $l$-$A$-$c:a.$ which is not $m$-convex, nor uniformly $A$-convex. Take e.g., $E = C_0(\mathbb{R}, \mathbb{R})$ of Example 2 and $F = C(\mathbb{R}, \mathbb{R})$ of Example 3.

Example 26. To have non commutative strictly real $l$-$A$-$c:a.$’s, take a non commutative strictly real $l.u$-$A$-$c:a.$ $E$ that is not $m$-convex and $F$ a non commutative (or even commutative) strictly real $m$-convex algebra which is not uniformly $A$-convex. Then $E \times F$ is a non commutative strictly real $l$-$A$-$c:a.$, which is not $m$-convex nor uniformly $A$-convex.

6. Locally uniformly $A$-convex algebras

The approach is structural. Let $(E, \tau)$ be a unital $l.u$-$A$-$c:a.$ As in the complex case, there is an algebra norm $\|\cdot\|_0$ stronger than $\tau$ (cf. [14], [15], [16]). If the latter is given by a family $(p_\lambda)_\lambda$ of seminorms (cf. Preliminaries), one may suppose that $p_\lambda(e) = 1$ for every $\lambda$, where $e$ is the unit element. Then one has $p_\lambda(x) \leq M(x)$, for every $\lambda$. Just put $\|x\|_0 = \sup\{ p_\lambda(x) : \lambda \}$. Moreover, $(E, \|\cdot\|_0)$ is a Banach algebra if $(E, \tau)$ is $M$-complete.

Proposition 27. Let $(E, (|.|)_A)$ be a strictly real $l.u$-$A$-$c:a.$

(i) If $(E, (|.|)_A)$ is unital and $M$-complete, then $E$ is commutative modulo its (Jacobson) radical $\text{Rad}(E)$.

(ii) If $(E, (|.|)_A)$ is non unital and complete, then $E$ is commutative modulo its (Jacobson) radical $\text{Rad}(E)$.
Proof. (i) If $E$ is unital, then $(E, \| \cdot \|_0)$ is a Banach algebra. So, here one uses directly the result of I. Kaplansky.

(ii) If $E$ is not unital, the unitization $E_1$ of $E$ a l.u.A.c.a., one can use Proposition 22.

Remark 28. The unitization of a l.u.A.c.a. is not always a l.u.A.c.a. (cf. [18]).

Concrete examples to which the previous proposition applies are the following.

Example 29. Let $C_b(R, R)$ be the algebra of real continuous bounded functions on the real field $R$ with the usual pointwise operations. Denote by $C_0^+(R, R)$ the strictly positive elements of $C_b(R, R)$. Consider the family $\{p_\varphi : \varphi \in C_0^+(R)\}$ of seminorms given by

$$p_\varphi(f) = \sup\{|f(x)\varphi(x)| : x \in \mathbb{R}\}; f \in C_b(\mathbb{R}).$$

They determine a locally convex topology $\beta$. The space $(C_b(R, R), \beta)$ is a complete locally convex algebra. It is not a l.m.c.a. [4]. But it is a l.u.A.c.a. with continuous multiplication.

Example 30. Consider the multiplier algebra $M(E)$ of a semisimple strictly real Banach algebra (see Section 3). Endow it with the strong operator topology, given by the family $(|.|_x)_x$ of seminorms defined by $|T|_x = \|T(x)\|$. As in the complex case [18], $(M(E), (|.|_x)_x)$ is a complete l.u.A.c.a., the associated norm $\|x\|_0$ of which is the operator norm.

To have non unital strictly real l.u.A.c.a.’s, one can use the following general construction.

Example 31. Take a non unital strictly real Banach algebra $E$. Put $E_1 = E$, $E_n = E \times \ldots \times E$ ($n$ times) and $F = \bigcup E_n$. Then clearly $F = \lim(E_n, \| \cdot \|_n)$, the transmission maps being the canonical injections $f_{nm} : E_m \longrightarrow E_n$, for $m \leq n$. Since $E_m$ is closed in $E_n$, one has $Sp(x) = Sp_{E_m}(x)$, for every $n$ such that $x \in E_n$. Thus $E$ is strictly real, as in Proposition 15.

We can use the idea of Remark 11, to obtain non commutative strictly l.u.A.c.a.’s.

Example 32. Consider a multiplier algebra $M(E)$ as in Section 3. Endow it with a multiplication $\times_T$ as in Remark 11, that is $P \times_T Q = PT(Q)$ and the strong topology $\beta$ given by the family $(|.|_x)_x$, $x \in E$, wher $|S|_x = \|S(x)\|$. One has

$$|P \times_T Q|_x = \|PT(Q)(x)\| = \|T(Q)P(x)\| \leq \|T(Q)\| |P|_x, \forall P.$$ 

And also

$$|P \times_T Q|_x = \|PT(Q)(x)\| \leq \|P\| \|T(Q)(x)\| = \|P\| \|Q(x)\| = \|P\| |Q|_x, \forall Q.$$ 

So $M(E)$ becomes a non commutative strictly real l.u.A.c.a.’s.
7. Locally uniformly convex algebras

Such complex algebras have been introduced in [21] (see also [19]). The definition remains the same (see Preliminaries). We will exhibit examples of l.u.c.a.‘s which are not uniformly A-convex. But examine first the subnormability of such algebras. A locally convex algebra \((E, \tau)\) is said to be subnormable if it can be endowed with a vector space norm which induces a topology stronger than \(\tau\). Exactly as in the complex case [21] one shows some basic facts.

**Proposition 33.** (i) A unital l.u.c.a. is subnormable. If moreover it is barrelled, then it is normable.

(ii) A non unital l.u.c.a. is subnormable if, and only if, its unitization \(E_1\) is also a l.u.c.a.

**Remark 34.** The norm in (i) of the previous proposition is a vector space but not necessarily an algebra norm (Example 36 below). This example shows also that, unlike the case of l.u.-A.c.a.‘s, not every element here is necessarily bounded.

Under an additional completeness condition, one can say more.

**Proposition 35.** ([21], [19]). Let \((E, \tau)\) be a unital and Mackey complete l.u.c.a. Then there is a Banach algebra norm stronger than \(\tau\) with the same bound structure.

The first example, adapted here to the real case, has been given in [21].

**Example 36.** Let \(\mathbb{R}[X]\) be the algebra of real polynomials and \((x_m)_m\) a sequence of real numbers such that \(|x_m| \to +\infty\). Endow it with the topology \(\tau\) given by the seminorms \(P \mapsto |P|_m = |P(x_m)|\). It becomes a unital commutative and metrizable l.m.c.a. Moreover it has a denumerable algebraic basis, so it is subnormable (cf. a Lemma in [7], p. 1039). Let \(\| \cdot \|\) be a vector space norm stronger than \(\tau\). Then, for every \(m\), there is a \(k_m > 1\) such that

\[ |PQ|_m = |P|_m |Q|_m \leq \|P\| k_m |Q|_m ; \forall Q. \]

To see the local uniform convexity of \(\mathbb{R}[X]\), consider the family \((|P|_m)_{m \cup (\alpha |_m)_{m, \alpha \geq 1}}\) of seminorms which also define the topology. It can not be uniformly A-convex, for otherwise we should have a stronger algebra norm \(\| \cdot \|_0\) than \(\tau\). But then the characters \(P \mapsto P(x_m)\) should be continuous for that norm, which contradicts \(|P(x_m)| \to +\infty\).

In the previous example, the algebra is not sequentially complete. Here we give an example where it is.

**Example 37.** Let \(E = C^1[0, 1]\) be the algebra of \(\mathbb{K}\)-valued \((\mathbb{K} = \mathbb{R}, \mathbb{C})\) continuous functions, on \([0, 1]\), with continuous derivative also at extreme points. Endow it with the topology \(\tau\) given by the seminorms \(\|f\|_\infty = \sup\{|f(t)| : 0 \leq t \leq 1\}\) and \(|f|_{K_d} = \sup\{|f'(t)| : t \in K_d\}\), where \(K_d\) runs over denumerable compact subsets of \([0, 1]\). Then \((E, \tau)\) is a unital commutative sequentially complete l.m.c.a.. It is a l.u.c.a.. Indeed,

\[ |fg|_{K_d} = \sup\{|f'(t)g(t) + f(t)g'(t)| : t \in K_d\} \leq M(f) \left[|g|_{K_d} + |g'|_{K_d}\right], \text{ with } M(f) = \max (\|f\|_\infty, \|f'\|_\infty). \]
It can not be a l.u.-A.c.a. for otherwise, there should be an algebra norm \( \| \cdot \|_0 \) stronger than \( \tau \), which is moreover the coarsest norm with such properties (see [15] or [16]). But then

\[ \tau_{\| \cdot \|_0} \leq \tau \leq \tau_{\| \cdot \|_0} \text{ and } \tau_{\| \cdot \|_0} \leq \tau_{\| \cdot \|_\infty}. \]

So \( \tau \) should be equivalent to \( \tau_{\| \cdot \|_0} \), which is not the case. Here the norm of Proposition 33 is \( \| \cdot \|_\infty + \| \cdot \|_\infty \), where \( \| f \|_\infty = \| f^* \|_\infty \).

**Example 38.** Consider \((L^\omega[0, 1], (\| \cdot \|_n)_n)\) the real Arens algebra [2], where \( \| f \|_n = \left( \int_0^1 |f(t)|^n \, dt \right)^{\frac{1}{n}} \), \( n \in \mathbb{N}^* \). It is known that \( C[0, 1] \) endowed with the induced topology is dense in \( L^\omega[0, 1] \). So here \( C[0, 1] \) is a l.u.-A-c.a. (hence a l.u.c.a.) the completion of which is an algebra. But, it is not of the same type, since as metrizable and complete (a \( B_0 \)-algebra) it should be normable (Proposition 7.1). So \((L^\omega[0, 1], (\| \cdot \|_n)_n)\) is a strictly real Fréchet l.c.a. It can not be A-convex, nor uniformly convex.

**Example 39.** \( C^1[0, 1] \times C_b(\mathbb{R}, \mathbb{R}) \) is a commutative strictly real l.u.c.a., which is not uniformly A-convex nor m-convex. It is a l-A.c.a.

Now using Proposition 35 and Kaplansky’s result, one obtains the following.

**Proposition 40.** A unital strictly real \( M \)-complete l.u.c.a. \((E, (\| \cdot \|_\lambda)_\lambda)\) is commutative modulo its (Jacobson) radical \( \text{Rad}(E) \).

We can use the idea of Remark 11, to obtain non commutative strictly real l.u.c.a.’s. Actually one gets real or complex algebras, which is a noticeable fact.

**Example 41.** Consider a multiplier \( M_T(E) \) as in Example 32, endowed with the multiplication \( \times_T \) and the strong topology \( \beta \) given by the family \((\| \cdot \|_x)_x, x \in E\), where \( \| S \|_x = \| S(x) \| \). It is a non commutative strictly l.u.-A-c.a.’s. Take the algebra \( C^1[0, 1] \) of Example 37. Then \( M_T(E) \times C^1[0, 1] \) is non commutative strictly l.u.c.a.. It is not a l.u.-A-c.a., but it is locally A-convex.

**Example 42.** One may also consider \( M_T(E) \times C_b(R, R) \) or even \( M_T(E) \times C^1[0, 1] \times C_b(R, R) \).

8. Conclusion

I. Kaplansky introduced strictly real Banach algebras, and showed that they are commutative modulo their Jacobson radical. In this paper, we extend this result to classes of locally convex (non normable) algebras. First, it is worthwhile to notice that the multiplier algebra \( M(E) \) of a strictly real Banach algebra is also strictly real. Kaplansky’s result is obtained for m-convex algebras, and a remark is made about Michaël problem. The same for A-convex algebras. Finally, we treat the case of uniformly convex algebras.

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