The structure of some 2-groups
and the capitulation problem for certain biquadratic fields

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Abstract. We study the capitulation problem for certain number fields \(K\) of degree 4 and we show how we can determine the structures of some 2-groups as an application of this study. Let \(K_1^{(2)}\) be the Hilbert 2-class field of \(K\), \(K_2^{(2)}\) be the Hilbert 2-class field of \(K_1^{(2)}\), \(C_{K,2}\) be the 2-component of the ideal class group of \(K\) and \(G_2\) the Galois group of \(K_2^{(2)}/K\). We suppose that \(C_{K,2}\) is of type \((2,2)\); then \(K_1^{(2)}\) contains three extensions \(F_i/K\), \(i = 1, 2, 3\). The aim of this paper is to study the capitulation of the 2-ideal classes in \(F_i\), \(i = 1, 2, 3\), and as an application of this study, to determine the structure of \(G_2\) and the structure of the 2-class group of the three fields \(F_i\), \(i = 1, 2, 3\), for the following cases:

(I) \(K = \mathbb{Q}(\sqrt{2q_1q_2}, i)\) where \(q_1\) and \(q_2\) are primes such that \(q_1 \equiv q_2 \equiv -1 \mod 4\). This is a case of biquadratic bicyclic number fields of \(\mathbb{Q}\).

(II) \(K = \mathbb{Q}(\sqrt{-pq(2 + \sqrt{2})})\) where \(p\) and \(q\) are primes such that \(p \equiv -q \equiv 5 \mod 8\). This is a case of quartic number fields of \(\mathbb{Q}\).

Key words: unit, class group, capitulation, Hilbert class field.

1. Intoduction

Let \(K\) be a number field of finite degree over \(\mathbb{Q}\) and \(C_K\) be the class group of \(K\). Let \(F\) be an unramified extension of \(K\) of finite degree and let \(O_F\) be its ring of integers. We say that an ideal \(\mathcal{A}\) (or the ideal class of \(\mathcal{A}\)) of \(K\) capitulates in \(F\) if it becomes principal in \(F\), i.e., if \(\mathcal{A}O_F\) is principal in \(F\). The Hilbert class field \(K_1\) of \(K\) is the maximal abelian unramified extension of \(K\). Let \(p\) be a prime number; the Hilbert \(p\)-class field \(K_1^{(p)}\) of \(K\) is the maximal abelian unramified
extension of $K$ such that $[K_1^{(p)} : K] = p^n$ for some integer $n$. The first important result on capitulation was conjectured by D. Hilbert and proved by E. Artin and P. Furtwängler. It deals with the case $F = K_1$.

**Theorem 1** (Principal ideal theorem). Let $K_1$ be the Hilbert class field of $K$, then every ideal of $K$ capitulates in $K_1$.

The principal ideal theorem was generalized by Tannaka and Terada to the next one. Let $K_0$ be a subfield of $K$ such that $K/K_0$ is abelian and let $(K/K_0)^*$ be the relative genus field of $K/K_0$.

**Theorem 2** (Tannaka–Terada). If $K/K_0$ is cyclic, then any ambiguous ideal class of $K/K_0$ is principal in $(K/K_0)^*$.

The case where $F/K$ is a cyclic extension of prime degree was studied by D. Hilbert in his Theorem 94:

**Theorem 3** (Theorem 94). Let $F/K$ be a cyclic extension of prime degree, then there exists at least one class (not trivial) in $K$ which capitulates in $F$.

We find in the proof of Theorem 94 this result:

Let $\sigma$ be a generator of the Galois group of $F/K$ and $N_{F/K}$ be the norm of $F/K$. Let $E_F$ be the unit group of the field $F$. Let $E_F^*$ be the group of units of norm 1 in $F/K$. Then the group of classes of $K$ which capitulates in $F$ is isomorphic to the quotient group $E_F^*/E_F^{1-\sigma} = H^1(G,F)$, the cohomology group of $G = \langle \sigma \rangle$ acting on the group $E_F$.

With this result and other results on cohomology, we have:

**Theorem 4** ([11]). Let $F/K$ be a cyclic unramified extension of prime degree, then the number of classes which capitulate in $F/K$ is equal to

$$[F : K][E_K : N_{F/K}(E_F)],$$

where $E_K$ (resp. $E_F$) is the unit group of $K$ (resp. $F$).

The case where $F/K$ is an abelian extension was treated by H. Suzuki who has proved Miyake’s conjecture: In an abelian extension $F/K$ the number of classes of $K$ which capitulates in $F$ is a multiple of $[F : K]$.

Moreover, H. Suzuki has proved the next theorem which is a generalization of the principal ideal theorem, the Hilbert theorem 94 and Tannaka-Terada’s principal ideal theorem:

**Theorem 5**. Let $K$ be a finite cyclic extension of an algebraic number field $K_0$ of finite degree, and let $F$ be an unramified extension of $K$ which is abelian over $K_0$. Then the number of the $G(K/K_0)$-invariant ideal classes of $K$ which become principal in $F$ is divisible by the degree $[F : K]$ of the extension $F/K$. 

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Frontiers in Science and Engineering
An International Journal Edited by Hassan II Academy of Science and Technology
Let $p$ be a prime number and let $K_1^{(p)}$ (resp. $K_2^{(p)}$) be the Hilbert $p$-class field of $K$ (resp. of $K_1^{(p)}$). If $L$ is a subfield of $K_1$ and $A$ is an ideal class of $K$ whose order is equal to $p^m$ for some integer $m$, then $A$ capitulates in $L$ if and only if $A$ capitulates in $L \cap K_1^{(p)}$. So we study only the capitulation of classes whose orders are equal to $p^m$ in the subfields of $K_1^{(p)}$, and since the capitulation problem is solved when $K_1^{(p)}/K$ is cyclic, we study only the cases where $K_1^{(p)}/K$ is not cyclic.

For more details see [18], [20], [1] and [21].

**Definition 6.** Let $F$ be a cyclic unramified extension of $K$, $C_F$ be its class field and $j$ the application of $C_K$ in $C_F$ that maps to the class of an ideal $\alpha$ of $K$, the class of the ideal generated by $\alpha$ in $F$. Then the extension $F/K$ is called:

- of type $(A)$ if and only if $\# \ker j \cap N_{F/K}(C_F) > 1$;
- of type $(B)$ if and only if $\# \ker j \cap N_{F/K}(C_F) = 1$.

**Proposition 7** ([14]). Let $G$ be a 2-group of finite order $2^m$ and $G'$ its derived subgroup. Then $G/G'$ is of type $(2, 2)$ if and only if $G$ is isomorphic to one of 2-groups:

$$
Q_m = \langle \sigma, \tau \rangle \quad o \quad \sigma^{2m-2} = \tau^2 = a, \quad a^2 = 1, \quad \tau^{-1} \sigma \tau = \sigma^{-1};
$$
$$
D_m = \langle \sigma, \tau \rangle \quad o \quad \sigma^{2m-1} = \tau^2 = 1, \quad \tau^{-1} \sigma \tau = \sigma^{-1};
$$
$$
S_m = \langle \sigma, \tau \rangle \quad o \quad \sigma^{2m-1} = \tau^2 = 1, \quad \tau^{-1} \sigma \tau = \sigma^{-1};
$$
$$
(2, 2) = \langle \sigma, \tau \rangle \quad o \quad \sigma^2 = \tau^2 = 1, \quad \tau^{-1} \sigma \tau = \sigma.
$$

Where $Q_m$ the quaternion group, $D_m$ the dihedral group, $S_m$ semi-dihedral group of order $2^m$ and $(2, 2)$ is an abelian group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Let $K$ be a number field such that the 2-component $C_{K,2}$ of $C_K$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $G_2$ be the Galois group of $K_2^{(2)}/K$. By class field theory, $Gal(K_1^{(2)}/K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $K_1^{(2)}$ contains three quadratic extensions of $K$ denoted by $F_1$, $F_2$, and $F_3$; precisely $F_1$ is the subfield of $K_2^{(2)}$ left fixed by the subgroup $\langle \sigma \rangle$, $F_2$ is the subfield of $K_2^{(2)}$ left fixed by the subgroup $\langle \sigma^2, \tau \rangle$ and $F_3$ is the subfield of $K_2^{(2)}$ left fixed by the subgroup $\langle \sigma^2, \sigma \tau \rangle$. Furthermore, if $G_2' \neq 1$, then $K_1^{(2)} \neq K_2^{(2)}$ and there exists a unique subgroup of $G_2$ of index 2; let $L$ be the subfield of $K_2^{(2)}$ left fixed by this subgroup and $j_i$ the mapping $j$ defined for $F = F_i$. Under these conditions, H. Kisilevsky, in [14], proved the following

**Theorem 8** ([14]). Assume that $C_{K,2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so we have

1. If $K_2^{(1)} = K_2^{(2)}$, then the fields $F_i$ are of type $(A)$, $\# \ker j_i = 4$ for $i = 1, 2, 3$ and $G_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$;

2. If $Gal(L/K) \simeq Q_3$, then the fields $F_i$ are of type $(A)$, $\# \ker j_i = 2$ for $i = 1, 2, 3$ and $G_2 \simeq Q_3$.
3. If \( \text{Gal}(L/K) \simeq D_3 \), then the fields \( F_2 \) and \( F_3 \) are of type (B) and \( \# \ker j_2 = \# \ker j_3 = 2 \). Moreover, if \( F_1 \) is of type (B) then \( \# \ker j_1 = 2 \) and \( G_2 \simeq S_m \). If \( F_1 \) is of type (A) and \( \# \ker j_1 = 2 \), then \( G_2 \simeq Q_m \). Finally if \( F_1 \) is of type (A) and \( \# \ker j_1 = 4 \), then \( G_2 \simeq D_m \).

**Corollary 9.** Let \( K \) be such that \( C_{K,2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Then we have three types of capitulation:

- **Type 1:** The four classes of \( C_{K,2} \) capitulate in each extension \( F_i \), \( i = 1, 2, 3 \).
- **Type 2:** The four classes of \( C_{K,2} \) capitulate only in one extension among the three extensions \( F_i \), \( i = 1, 2, 3 \). In this case the group \( G_2 \) is dihedral.
- **Type 3:** Only two classes capitulate in each extension \( F_i \), \( i = 1, 2, 3 \). In this case the group \( G_2 \) is semi-dihedral or quaternionic.

**Remark 10.** The 2-class group of \( F_1 \) is cyclic. The 2-class groups of \( F_2 \) and \( F_3 \) are cyclic in the cases 1 and 2 of the theorem 8 and are of type (2, 2) in the third case.

## 2. Units of some number fields

Let \( d_1 \), \( d_2 \) be coprime integers, which are square-free, \( d_3 = d_1d_2 \), \( \varepsilon_1 \) (resp. \( \varepsilon_2, \varepsilon_3 \)) the fundamental unit of \( \mathbb{Q}(\sqrt{d_1}) \) (resp. \( \mathbb{Q}(\sqrt{d_2}), \mathbb{Q}(\sqrt{d_3}) \)), \( K_0 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}), Q_{K_0} \) the Hasse unit index of \( K_0 \) and \( N_i \) the norm of \( K_0/Q_{i}^{(2)} \) with \( i \in \{1, 2, 3\} \).

From [15], we know that a fundamental system of units of \( K_0 \) is one of the following:

- \( \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \);
- \( \{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\} \) \( \left( N_2(\varepsilon_3) = 1 \right) \);
- \( \{\sqrt{\varepsilon_1}\varepsilon_2, \varepsilon_2, \varepsilon_3\} \) \( \left( N_3(\varepsilon_1) = N_3(\varepsilon_2) = 1 \right) \);
- \( \{\varepsilon_1, \sqrt{\varepsilon_2}, \sqrt{\varepsilon_3}\} \) \( \left( N_1(\varepsilon_2) = N_1(\varepsilon_3) = 1 \right) \);
- \( \{\sqrt{\varepsilon_1}\varepsilon_2, \sqrt{\varepsilon_2}\varepsilon_3, \sqrt{\varepsilon_1}\varepsilon_3\} \) \( \left( N_2(\varepsilon_3) = N_3(\varepsilon_j) = 1, j = 1, 2 \right) \);
- \( \{\sqrt{\varepsilon_1}\varepsilon_2, \sqrt{\varepsilon_3}, \varepsilon_2, \varepsilon_3\} \) \( \left( N_3(\varepsilon_1) = N_3(\varepsilon_2) = N_3(\varepsilon_3) = \pm 1 \right) \).

**Proposition 11** ([2]). Let \( K_0 \) a real number field, \( F = K_0(\sqrt{-1}) \) a quadratic extension of \( K_0 \), abelian and finite over \( \mathbb{Q} \) and \( \{\varepsilon_1, \varepsilon_2, ....., \varepsilon_r\} \) be a fundamental system of units of \( K_0 \) (whose units are all positive). Then we have:

1. If there is a unit of \( K_0 \) of the form \( \varepsilon = \varepsilon_1^{j_1}\varepsilon_2^{j_2}....\varepsilon_{r-1}^{j_{r-1}}\varepsilon_r \) where \( j_k \in \{0, 1\} \), such that \( (2 + \mu_m)\varepsilon \) is a square in \( K_0 \), then \( \{\varepsilon_1, \varepsilon_2, ....., \varepsilon_{r-1}, \sqrt{\varepsilon} \} \) is a fundamental system of units of \( F \);

2. Otherwise \( \{\varepsilon_1, \varepsilon_2, ....., \varepsilon_r\} \) is a fundamental system of units of \( F \).

**Proposition 12** ([2]). Let \( K_0 \) a number field, abelian real and \( \beta \) an algebraic integer in \( K_0 \), completely positive, without square factors. Assume that \( F = K_0(\sqrt{-1}) \) is a quadratic extension of \( K_0 \), abelian over \( \mathbb{Q} \) and \( i = \sqrt{-1} \) doesn’t belong to \( F \). Let \( \{\varepsilon_1, \varepsilon_2, ....., \varepsilon_r\} \) be a fundamental system of units of \( K_0 \). We choose, without limiting the generality, units \( \varepsilon_j \) positive. Then we have:

1. If there is a unit of \( K_0 \) such that \( \varepsilon = \varepsilon_1^{j_1}\varepsilon_2^{j_2}....\varepsilon_{r-1}^{j_{r-1}}\varepsilon_r \) (close to a permutation), where the \( j_k \in \{0, 1\} \), such that \( \beta\varepsilon \) is a square in \( K_0 \), then \( \{\varepsilon_1, \varepsilon_2, ....., \varepsilon_{r-1}, \sqrt{-\varepsilon} \} \) is a fundamental system of units of \( F \);
Lemma 16. Therefore, if $\varepsilon^2 \equiv 1 \pmod{d}$ are congruent to $q$ of this paragraph, then $\{\varepsilon\}$ is a fundamental system of units of $L$.

Corollary 13 ([7]). Let $L = \mathbb{Q}(\sqrt{-n\varepsilon\sqrt{d}})$ be a cyclic extension of degree 4 over $\mathbb{Q}$, where $\varepsilon$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ with $d$ a square-free integer and $n$ an integer, then $\{\varepsilon\}$ is a fundamental system of units of $L$.

Lemma 14 ([3], Theorem 14). Let $p$ and $q$ be odd prime numbers such as $q \equiv -1 \pmod{4}$, $K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$, $F = K_0(i)$ and $\varepsilon_1$ (resp. $\varepsilon_2$, $\varepsilon_3$) be the fundamental unit of $\mathbb{Q}(\sqrt{2})$ (resp. $\mathbb{Q}(\sqrt{pq})$, $\mathbb{Q}(\sqrt{2pq})$). Assume that $2\varepsilon_3$ is not a square in $\mathbb{Q}(\sqrt{pq})$ and $\varepsilon_2 = x + y\sqrt{pq}$ with $(x, y) \in \mathbb{Z}^2$. Then we have:

(i) If $x \pm 1$ is a square in $\mathbb{N}$, then $\{\varepsilon_1, \sqrt{\varepsilon_2}, \sqrt{\varepsilon_3}\}$ is a fundamental system of units of $K_0$ and of $F$.

(ii) Otherwise, $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\}$ is a fundamental system of units of $K_0$ and of $F$.

Lemma 15 ([3], Theorem 14). We keep the notations of the previous lemma and assume that $p$ and $q$ are congruent to $-1$ modulo 4, and that $2\varepsilon_3$ is not a square in $\mathbb{Q}(\sqrt{pq})$ and $\varepsilon_2 = x + y\sqrt{pq}$ with $x$ and $y$ be two odd integers. Then $K_0$ and $F$ have the same fundamental system of units.

Lemma 16. Let $p$, $q$ be odd prime numbers, $K_0 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$, $\varepsilon_1$ (resp. $\varepsilon_2$, $\varepsilon_3$) the fundamental unit of $\mathbb{Q}(\sqrt{q})$ (resp. $\mathbb{Q}(\sqrt{2p})$, $\mathbb{Q}(\sqrt{2pq})$). Assume that all units $\varepsilon_i$ are of norm 1 and $2\varepsilon_3$ is not a square in $\mathbb{Q}(\sqrt{2pq})$. We set $\varepsilon_3 = x + y\sqrt{2pq}$. Then a fundamental system of units of $K_0$ is

(i) $\{\sqrt{\varepsilon_1\varepsilon_3}, \sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_2\varepsilon_3}\}$ if $2p(x \pm 1)$ is a square in $\mathbb{N}$.

(ii) $\{\varepsilon_1, \sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_3}\}$ if $2q(x \pm 1)$ is a square in $\mathbb{N}$.

Proof. Let $\varepsilon_3 = x + y\sqrt{2pq}$ such that $(x - 1)(x + 1) = 2pqy^2$. We know that $2\varepsilon_3$ is a square in $\mathbb{Q}(\sqrt{2pq})$ if and only if $x \pm 1$ is a square in $\mathbb{N}$.

- If $2(x \pm 1)$ is a square in $\mathbb{N}$, then there exists $(y_1, y_2) \in \mathbb{Z}^2$ such that
  \[
  \begin{cases}
  x \pm 1 = 2y_1^2 \\
  x \mp 1 = pqy_2
  \end{cases}
  \quad \text{and} \quad
  \varepsilon_3 = \frac{1}{2}(y_2\sqrt{2pq} + 2y_1) \in \mathbb{Q}(\sqrt{2pq}).
  \]
  This is contrary to the fact that $\varepsilon_3$ is the fundamental unit of $\mathbb{Q}(\sqrt{2pq})$. Following $2p(x \pm 1)$ or $2q(x \pm 1)$ is a square in $\mathbb{N}$.

- If $2p(x \pm 1)$ is a square in $\mathbb{N}$, then $q(x \mp 1)$ is a square in $\mathbb{N}$ and there exists $(y_1, y_2) \in \mathbb{Z}^2$ such that
  \[
  \begin{cases}
  x \pm 1 = 2y_1^2 \\
  x \mp 1 = pqy_2
  \end{cases}
  \quad \text{and} \quad
  \varepsilon_3 = \frac{1}{2}(y_2\sqrt{2pq} + 2y_1) \in \mathbb{Q}(\sqrt{2pq}).
  \]

Therefore, if $2p(x \pm 1)$ is a square in $\mathbb{N}$, then $2\varepsilon_3$ is a square in $K_0$ and if $2q(x \pm 1)$ is a square in $\mathbb{N}$, then $\varepsilon_3$ is a square in $K_0$. On the other hand, after the two previous lemmas, $2\varepsilon_1$ and $2\varepsilon_2$ are squares in $K_0$. Hence $\varepsilon_1\varepsilon_2$ is a square in $K_0$. Similarly, if $2\varepsilon_3$ is a square in $K_0$, then $\varepsilon_1\varepsilon_3$ and $\varepsilon_2\varepsilon_3$ are squares in $K_0$ and thus is a fundamental system of units of $K_0$. In the case where $\varepsilon_3$ is a square in $K_0$, the unit $\varepsilon_1\varepsilon_2\varepsilon_3$ is a square in $K_0$ and according to the results of [15] recalled the beginning of this paragraph, $\{\varepsilon_1, \sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_3}\}$ is a fundamental system of units of $K_0$. \[\square\]
Lemma 17. With the conditions of Lemma 16. Let \( F = K_0(i) \), then

(i) if \( 2p(x \pm 1) \) is a square in \( \mathbb{N} \), then \( \left\{ \sqrt{\varepsilon_1 \varepsilon_2}, \sqrt{\varepsilon_3}, \sqrt{i \varepsilon_2 \varepsilon_3} \right\} \) is a fundamental system of units of \( F \);

(ii) if \( 2q(x \pm 1) \) is a square in \( \mathbb{N} \), then \( \left\{ \sqrt{\varepsilon_1 \varepsilon_2}, \sqrt{\varepsilon_3}, \sqrt{i \varepsilon_2} \right\} \) is a fundamental system of units of \( F \).

Proof. (i) Let \( \varepsilon = \sqrt{\varepsilon_1 \varepsilon_2} \sqrt{\varepsilon_1 \varepsilon_3} \sqrt{\varepsilon_2 \varepsilon_3} = \varepsilon_1 \left( \sqrt{\varepsilon_2 \varepsilon_3} \right)^2 \). We know from Proposition 11 that \( \sqrt{\varepsilon} \in F \) if and only if \( \sqrt{2 \varepsilon} \in K_0 \). Since \( 2 \varepsilon_1 \) is a square in \( K_0 \), then \( 2 \varepsilon \) is a square in \( K_0 \). A fundamental system of units of \( K_0 \) is given by the preceding lemma. Hence, by Proposition 11, \( \left\{ \sqrt{\varepsilon_1 \varepsilon_2}, \sqrt{\varepsilon_1 \varepsilon_3}, \sqrt{i \varepsilon_1 \varepsilon_2 \varepsilon_3} \right\} \) is a fundamental system of units of \( F \).

(ii) Let \( \varepsilon = \varepsilon_2 \). We know from Lemma 16 that \( 2 \varepsilon_2 \) is a square in \( K_0 \) and Lemma 17 gives us a fundamental system of units of \( K_0 \). So while using Proposition 11, we find that \( \left\{ \sqrt{i \varepsilon_1 \varepsilon_2}, \sqrt{\varepsilon_3}, \sqrt{i \varepsilon_2} \right\} \) is a fundamental system of units of \( F \).

\[ \square \]

Theorem 18 ([6]). Let \( K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{p}), \varepsilon_0 \) (resp. \( \varepsilon_2, \varepsilon_3 \)) the fundamental unit of \( \mathbb{Q}(\sqrt{2}) \) (resp. \( \mathbb{Q}(\sqrt{p}) \), \( \mathbb{Q}(\sqrt{2p}) \)) and \( F = K_0(\sqrt{1 - m \varepsilon_0 \sqrt{2}}) \) where \( m \) is an odd integer. Then,

1) if \( \varepsilon_3 \) has norm 1, then \( \left\{ \varepsilon_0, \varepsilon_2, \sqrt{\varepsilon_3} \right\} \) is a fundamental system of units of \( K_0 \) and of \( F \);

2) otherwise, \( \left\{ \sqrt{\varepsilon_0 \varepsilon_2 \varepsilon_3}, \varepsilon_2, \varepsilon_3 \right\} \) is a fundamental system of units of \( K_0 \) and of \( F \).

Theorem 19. Let \( L = \mathbb{Q}(\sqrt{2} + \sqrt{2}) \) and \( F = L(\sqrt{1 - m}) \) where \( m \) is an odd integer and square-free. Then \( \left\{ \xi_3, \xi_5, \xi_7 \right\} \) is a fundamental system of units of \( L \) and of \( F \) where \( \xi_3 = 1 + \sqrt{2 + \sqrt{2}} \), \( \xi_5 = 1 + \sqrt{2 + \sqrt{2} + \sqrt{2}} \) and \( \xi_7 = 1 + \sqrt{2 + \sqrt{2} + \sqrt{2} + \sqrt{2}} \).

Proof. It is known in the literature that \( \left\{ \xi_3, \xi_5, \xi_7 \right\} \) is a fundamental system of units of \( L \) (see eg [24], p. 144-145). Since \( m \xi \) is not a square in \( L \), for \( \xi = \xi_1^{j_1} \xi_2^{j_2} \xi_3^{j_3} \) where \( \{\xi_1, \xi_2, \xi_3\} = \{\xi_3, \xi_5, \xi_7\} \) and \( j_1, j_2 \in \{0, 1\} \), then, according to the proposition 12, we find that the system \( \{\xi_3, \xi_5, \xi_7\} \) is a fundamental system of units of \( F \).

\[ \square \]

3. Case where \( K \) is a biquadratic bicyclic number field of \( \mathbb{Q} \)

Let \( K \) be the Hasse unit index of \( K \) and let \( C_{K,2} \) be the 2-component of the class group of \( K \). Let \( K^{(w)} \) be the genus field of \( K \). In this section, we suppose that \( K = \mathbb{Q}(\sqrt{2q_1q_2}, i) \) where \( q_1 \equiv q_2 \equiv -1 \mod 4, \left( \frac{q_1}{q_2} \right) = -1, \left( \frac{2}{q_1} \right) = -1, \left( \frac{2}{q_2} \right) = 1 \) and \( Q = 1 \), in which case, the group \( C_{K,2} \) is of type \( (2, 2) \) and \( K^{(w)} = K_1^{(2)} = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2}, \sqrt{2}, i) \). So \( K^{(2)}_1 \) contains three extensions \( F_i/K, i = 1, 2, 3 \). The aim of this section is to study the capitulation of the 2-ideal classes in \( F_i, i = 1, 2, 3 \), and to determine the structure of \( G_2 \). So we have:

\( F'_1 = K(\sqrt{q_1}) = \mathbb{Q}(\sqrt{q_1}, \sqrt{2q_2}, i), F'_2 = K(\sqrt{q_2}) = \mathbb{Q}(\sqrt{q_2}, \sqrt{2q_1}, i) \) and \( F'_3 = \sqrt{q_2}, i). \)
Theorem 20. Let \( K = \mathbb{Q}(\sqrt{2q_1q_2}, i) \) with \( \frac{q_1}{q_2} = \frac{q_2}{q_1} = \frac{2}{q_1} = \frac{2}{q_2} = 1 \) and the unit index \( Q \) of \( \mathbb{Q}(\sqrt{2q_1q_2}) \) in \( K \) is equal to 1. Then only two classes of \( K \) capitulate in each extension \( F'_i, i = 1, 2, 3 \) and the group \( G_2 \) is semi-dihedral or quaternionic.

Proof. To prove this theorem we will use the fact that in an unramified extension \( F/K \) of degree 2 the number of classes that capitulated in \( F \) is equal to \( 2 | E_K : N_{F/K}(E_F) | \). Let \( \varepsilon_3 \) be the fundamental unit of \( \mathbb{Q}(\sqrt{2q_1q_2}) \), then \( E_K \), the unit group of \( K \), is generated by \( \varepsilon_3 \) and the complex number \( i \), so we must compute \( N_{F/K}(E_F) \) in each case of \( F( F = F'_i, i = 1, 2, 3) \).

On the other hand, let \( \varepsilon_2 = x + y \sqrt{q_1q_2} \) be the fundamental unit of \( \mathbb{Q}(\sqrt{q_1q_2}) \). Then

a) If \( x \) and \( y \) are integers, then exactly two classes of \( C_{K,2} \) capitulate in \( F'_1 \). It is the same for \( F'_2 \), while in \( F'_3 \) the 4 classes capitulate if and only if \( x \pm 1 \) is a square in \( \mathbb{N} \) (a fundamental system of \( F'_1 \) or \( F'_2 \) is given by lemma 2.3 and lemma 2.4 and for \( F'_3 \) is given by lemma 2.1 and lemma 2.2). Let us show that \( x \pm 1 \) is never a square of \( \mathbb{N} \).

Since \( q_1q_2 \equiv 1 \) mod 4, then \( x \) is odd. Therefore, the greatest common divisor of \( x - 1 \) and \( x + 1 \) is equal to two. Moreover, using the equation \( (x - 1)(x + 1) = q_1q_2y^2 \), we deduce that the exponent of 2 in the decomposition of \( x - 1 \) and \( x + 1 \) into prime factors is odd. It follows that \( x \pm 1 \) can not be a square in \( \mathbb{N} \). Thus, our result is proved.

b) If \( x \) and \( y \) are half-integers. Then in each extension \( F'_i, (i \in \{1, 2, 3\}) \), exactly two classes of \( K \) capitulate. Indeed, in this case a fundamental system of units of \( K_3 \) and \( F'_3 \) is \( \{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2 \varepsilon_3}\} \).

Then we have that \( N(E_{F'_3}) = E_K \). As a result, there are two classes that capitulated in \( F'_3 \). It is the same for the extensions \( F'_1 \) and \( F'_2 \).

Then only two classes of \( K \) capitulate in each extension \( F'_i, i = 1, 2, 3 \). Therefore, by theorem 8 and corollary 9, \( G_2 \) is the quaternionic group or semi-dihedral group.

Theorem 21. Let \( K = \mathbb{Q}(\sqrt{2q_1q_2}, i) \) with \( \frac{q_1}{q_2} = \frac{q_2}{q_1} = \frac{2}{q_1} = \frac{2}{q_2} = 1 \) and the unit index \( Q \) of \( \mathbb{Q}(\sqrt{2q_1q_2}) \) in \( K \) is equal to 1. Let \( F'_1 = K(\sqrt{q_1}), F'_2 = K(\sqrt{q_2}) \) and \( F'_3 = K(\sqrt{2}) \), then the 2-class groups of \( F'_1 \) and \( F'_3 \) are of type \( (2, 2) \) and the 2-class groups of \( F'_2 \) is cyclic.

Proof. Using Wada’s formula on class number of multiquadratic fields ([23]) and Kaplan’s results on the 2-part of class number of quadratic fields ([13]) then we have:
- \( h(q_1), h(q_2), h(2q_1), h(2q_2), h(-q_1), h(-q_2) \) et \( h(q_1q_2) \) are odd;
- the 2-part of \( h(-2q_2) \) is equal to 2, the 2-part of \( h(-2q_1) \) is divided by 4 and is divided by 8 if and only if \( q_1 \equiv 15 \) mod 16, and the 2-part of \( h(-q_1q_2) \) is equal to 4;
- let \( Q' \) be the index of the product of the units groups of all the quadratic fields which are subsets of \( F \) in the unit group of \( F \) ( if \( F = F'_i \), then \( Q' = [E_{F'_i} : E_{-1}E_{q_1}E_{-q_1}E_{2q_2}E_{-2q_2}E_{2q_1q_2}E_{-2q_1q_2} \])

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The 2-class group of $F_i'$ is cyclic of order $2$, and the 2-class number of $F_i'$ is equal to $1$. Then the group $G_2$ is semi-dihedral.

**Proof.** The 2-class group of $K$ can be generated by classes of prime ideals of $K$ laying above the ramified primes in $K/\mathbb{Q}$. But in our case this is not possible, we have only one non trivial class generated by the prime ideal $I_0$ above the prime $2$ ($2O_K = I_0^3$). We choose the second generator as the following:

Let $l$ be a prime such that $(\frac{q_1}{l}) = (\frac{q_2}{l}) = -1$ and $(\frac{2}{l}) = (\frac{-1}{l}) = 1$. The prime $l$ exists (see [Sim-95]) and splits completely in $K/\mathbb{Q}$, then there exist some ideals $I_1, I_2, I_3$ and $I_4$ of $K$ such that $lO_K = I_1I_2I_3I_4$. The ideal $I_1$ is inert in $F_2'/K$, so by the Artin Reciprocity theorem we prove that $I_1$ is not principal and, if $m$ is the odd part of the class number of $K$, $I_1^m$ is also not principal.

Then $C_{K,2}$ is generated by the class of $I_0$ and the class of $I_1^m$.

The ideal $I_1$ is inert in $F_2'/K$. Also the ideal $I_0$ is inert in $F_2'/K$. So we have:

- the ideal class of $I_1^m$ isn’t norm in $F_2'/K$.
- the ideal class of $I_0$ isn’t norm in $F_2'/K$.
- the ideal class of $I_0I_1^m$ isn’t norm in $F_2'/K$.

From these remarks and with using theorem 8, we obtain that one class from the classes of $I_0$, $I_1^m$ or $I_0I_1^m$ capitulates in $F_2'/K$ and it isn’t norm from $F_2'$; so $F_2'$ is of type (B). The field $F_2'$ of our case is the field $F_1$ in theorem 8, so by this theorem the group $G_2$ is semi-dihedral.

**Example 23.** Let $q_1 = 7$, $q_2 = 3$ and $d = 2q_1q_2 = 42$.

The 2-class group of of $K = \mathbb{Q}(\sqrt{42}, i)$ is of type $(2, 2)$, $F_i' = K(\sqrt{7})$, $F_2' = K(\sqrt{3})$ and $F_3' = K(\sqrt{2})$. Then only two classes of $K$ capitulate in each extension $F_i'$, $i = 1, 2, 3$. Moreover, in this case, the 2-class group of $F_2'$ is cyclic of order 8 and $G_2$ is semi-dihedral of order 16.

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A. Azizi and M. Talbi  
*The structure of some 2-groups*
4. Case where $K$ is a quartic number field of $\mathbb{Q}$

In this section, we suppose that $K = \mathbb{Q}(\sqrt{-pq(2 + \sqrt{2})})$ where $p$ and $q$ are primes such that $p \equiv -q \equiv 5 \pmod{8}$, in which case, the group $C_{K,2}$ is of type $(2, 2)$ (see [8]), by using the results of [12] we show that the genus field of $K$ is $K' = K^{(2)} = K(\sqrt{p}, \sqrt{-q})$. So $K^{(2)}$ contains three extensions $F_i'/K$, $i = 1, 2, 3$. The aim of this section is to study the capitulation of the 2-ideal classes in $F_i'$, $i = 1, 2, 3$, and to determine the structure of $G_2$. So we have: $F_1' = K(\sqrt{p})$, $F_2' = K(\sqrt{-q})$ and $F_3' = K(\sqrt{-pq})$.

Proposition 24. Let $L/M$ be a CM-extension, $\Delta$ its Galois group, and $v_0$ a place of $M$ such that the following conditions hold:

1) $C_{M,2} = 0$;

2) $L$ and $M$ have the same units (hence $\sqrt{-1} \notin L$);

3) $L = M(\sqrt{\beta})$ with $\beta \in M$, and $2 \nmid v_0(\beta)$.

then the natural map

$$\bigoplus_{v \in \text{Ram}(L/M), v \neq v_0} \mathbb{Z}/2\mathbb{Z} \rightarrow C_{L,2}$$

which, to a ramified place of $M$ associates the class of its square root in $L$, is injective.

Proof. Let $I_{2,L}$ (resp. $I_{2,M}$) the group of fractional ideals of $L$ tensored with $\mathbb{Z}_2$ (resp. $M$), and $P_{2,L}$ (resp. $P_{2,M}$) its sub $\mathbb{Z}_2$-module generated by the principal ideals. Let us analyze the image of the natural map $(I_{2,L})^{\Delta} \rightarrow (C_{L,2})^{\Delta}$. Thanks to (1), it factors into a map

$$\phi : (I_{2,L})^{\Delta}/I_{2,M} \rightarrow (C_{L,2})^{\Delta}$$

(here we embed $I_{2,M}$ into $I_{2,L}$ via the obvious map) which kernel is

$$(P_{2,L})^{\Delta}/I_{2,M} = (P_{2,L})^{\Delta}/P_{2,M} \simeq \text{coker}((L^\times)^{\Delta} \rightarrow (P_{2,L})^{\Delta}) \simeq H^1(\Delta, O_L^\times).$$

Thanks to (2), we conclude that $\ker \phi$ is of order 2. Now, because of (3), $(\sqrt{\beta})$ gives a non trivial element of $\ker \phi$, hence generates it. The claimed result followed by explicitly writing $\phi$ as

$$\phi : \bigoplus_{v \in \text{Ram}(L/M), v \neq v_0} \mathbb{Z}/2\mathbb{Z} \rightarrow (C_{L,2})^{\Delta}$$

Indeed, $(\sqrt{\beta})$ corresponds to the element $\bigoplus v(\beta)$ in the left hand group so that

$$\ker \phi \cap \bigoplus_{v \in \text{Ram}(L/M), v \neq v_0} \mathbb{Z}/2\mathbb{Z} = 0.$$
Corollary 25. Let $K = \mathbb{Q}(\sqrt[2]{-pq(2 + \sqrt{2})})$ where $p$ and $q$ are primes such that $p \equiv -q \equiv 5 \mod 8$, $\mathcal{P}$ the prime ideal of $K$ above $p$ and $\mathcal{Q}$ that above $q$. Then the class of $\mathcal{P}$ (resp. $\mathcal{Q}$, $\mathcal{PQ}$) has order 2, $C_{K,2}$ is generated by the classes of $\mathcal{P}$ and of $\mathcal{Q}$. Also $\mathcal{P}$ capitulated in $F_1'$, $\mathcal{Q}$ capitulated in $F_2'$ and $\mathcal{PQ}$ capitulated in $F_3'$.

**Proof.** By the proposition 24, we find that $C_{K,2}$ is generated by the classes of $\mathcal{P}$ and $\mathcal{Q}$. To show that $\mathcal{P}$ capitulates in $K(\sqrt{p})$, it suffices to see that $\sqrt{p} \in K(\sqrt{p})$ and $(\sqrt{p}^2) = (p)$ in $K(\sqrt{p})$, so $\mathcal{P}$ capitulates in $F_1' = K(\sqrt{p})$ and even $\mathcal{Q}$ capitulated in $F_2' = K(\sqrt{-q})$ and $\mathcal{PQ}$ capitulated in $F_3' = K(\sqrt{-pq})$. □

Proposition 26 ([10]). Let $M$ a number field contains the $m$-th roots of unity, $L$ a finite extension of $M$, $\alpha \in M^*$ and $\beta \in L^*$. We denote by $P$ a prime ideal of $M$ and $\mathcal{P}$ a prime ideal of $L$ above $P$. Then

$$\prod_P \left( \frac{\beta, \alpha}{\mathcal{P}} \right)_m = \left( \frac{N_{L/M}(\beta), \alpha}{P} \right)_m,$$

where the product is taken over all prime ideals of $L$ which are over $P$.

Theorem 27. Let $K = \mathbb{Q}(\sqrt[2]{-pq(2 + \sqrt{2})})$ where $p$ and $q$ are primes such that $p \equiv -q \equiv 5 \mod 8$, then in each extension $F_i'$, $i \in \{1, 2, 3\}$, there are exactly two classes of $C_{K,2}$ which capitulated and the group $G_2$ is quaternionic of order $2^m$ with $m > 3$.

**Proof.** Let $\varepsilon_0$ (resp. $\varepsilon_2, \varepsilon_3$) the fundamental unit of $\mathbb{Q}(\sqrt{2})$ (resp. $\mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{2q})$), $\mathcal{P}$ the prime ideal of $K$ above $p$, $\mathcal{Q}$ that above $q$, then, by Theorem 18, $\{\sqrt{\varepsilon_0}, \varepsilon_2, \varepsilon_3\}$ is a fundamental system of units of $F_i'$.

As $N_{F_i'/K}(\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3^3}) = \pm\varepsilon_1$ and $N_{F_i'/K}(\varepsilon_2) = N_{F_i'/K}(\varepsilon_3) = -1$, then $E_K = N_{F_i'/K}(E_{F_i'})$. Using the Theorem 4, we find that two Classes only of $C_{K,2}$ caputlating in $F_i'$, namely the class of $\mathcal{P}$ and its square.

By Theorem 19, $\{\xi_3, \xi_5, \xi_7\}$ is a fundamental system of units of $F_3'$. Since $N_{F_3'/K}(\xi_7) = -1$ and $N_{F_3'/K}(\xi_5) = N_{F_3'/K}(\xi_3) = \varepsilon_1$, then $E_K = N_{F_3'/K}(E_{F_3'})$. By the theorem 4, we find that two classes only of $C_{K,2}$ caputlating in $F_3'$, namely the class of $\mathcal{P}$ and its square.

The extension $F_1'/K$ and $F_2'/K$ are of type $B$ and the extension $F_3'/K$ is of type $A$. Indeed, let $K' = \mathbb{Q}(\sqrt[2]{-q(2 + \sqrt{2})})$, then we have $KK' = F_1'$ and as $N_{K'/\mathbb{Q}(\sqrt{2})}(\mathcal{P}) = p$ and $p$ is unramified in $K'/\mathbb{Q}(\sqrt{2})$, then $p$ is inert in $K'/\mathbb{Q}(\sqrt{2})$. We find that $p$ is inert in $K'/\mathbb{Q}(\sqrt{2})$, (Translation theorem). For this we compute the norm residue symbol $\left( \frac{p, -q\varepsilon_0\sqrt{2}}{p} \right)$. It has $p \in \mathbb{Q}$ is inert in $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $-q\varepsilon_0\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, so using the proposition 26, we find

$$\left( \frac{p, -q\varepsilon_0\sqrt{2}}{p} \right) = \left( \frac{p, N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(-q\varepsilon_0\sqrt{2})}{p} \right) = \left( \frac{p, 2q^2}{p} \right) = \left( \frac{2}{p} \right) = -1,$$

thus $p$ is inert in $K'/\mathbb{Q}(\sqrt{2})$, which gives that $\mathcal{P}$ is inert in $F_1'/K$ and since $[\mathcal{P}]$ is the only nontrivial class of $C_{K,2}$ capitulating into $F_1'/K$, then $F_1'/K$ is of type $B$. Similarly we show that $Q$ is inert.
in $F'_3/K$, which also gives that $F'_3/K$ is of type $(B)$, consequently, by theorem 8, two classes only of $C_{K,2}$ capitulate in $F'_3$. For $F'_3$, let $L = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$, then we have $KL = F'_3$, $N_{K'/\mathbb{Q}(\sqrt{2})}(P) = p$ and $p$ is unramified in $L/\mathbb{Q}(\sqrt{2})$, thus to show that $P$ is inert in $F'_3/K$, it suffices to show that $p$ is inert in $L/\mathbb{Q}(\sqrt{2})$, for this we compute the norm residue symbol $\left( \frac{p, \sqrt{2}}{p} \right)$. Using the proposition 26, we have $p \in \mathbb{Q}$ is inert in $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\varepsilon_0 \sqrt{2} \in \mathbb{Q}(\sqrt{2})$, so

$$\left( \frac{p, \varepsilon_0 \sqrt{2}}{p} \right) = \left( \frac{p, N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\varepsilon_0 \sqrt{2})}{p} \right) = \left( \frac{p, \varepsilon_0 \sqrt{2}}{p} \right) = \left( \frac{2}{p} \right) = -1,$$

thus $p$ is inert in $L/\mathbb{Q}(\sqrt{2})$, which gives that $P$ is inert in $F'_3/K$, similarly one shows that $Q$ is inert in $F'_3/K$. As $PQ$ capitulated in $F'_3$, then by applying the reciprocity law of Artin, we find that $F'_3/K$ is of type $(A)$. Consequently, using theorem 8, the group $G_2$ is isomorphic to $Q_m$ with $(m > 3)$.

**Example 28.** Let $K = \mathbb{Q}(\sqrt{-55(2 + \sqrt{2})})$, $F'_1 = K(\sqrt{5}), F'_2 = K(\sqrt{-11})$ and $F'_3 = K(\sqrt{-55})$, then by theorem 27 in each extension $F'_i$, $i \in \{1, 2, 3\}$, there are exactly two classes of $C_{K,2}$ who capitulated and the group $G_2$ is quaternionic of order $2^m$ with $m > 3$.

**Remark 29.** Let $K = \mathbb{Q}(\sqrt{-pq(2 + \sqrt{2})})$ and $K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{-pq})$ where $p$ and $q$ are primes such that $p \equiv -q \equiv 5$ mod 8, then $\# G_2 = 4 h_2(K_0)$.

**Proof.** We have $K_2^{(1)}/F'_3$ unramified extension and as $F'_3/K$ is of type $(A)$, then, according to [14], $C_{F'_3,2}$ is cyclic, so $F'_3$ and $K_2^{(2)}$ has even Hilbert 2-class field, namely $K_2^{(2)}$, so $\# G_2 = 2 h_2(F'_3)$. Moreover $F'_3/\mathbb{Q}(\sqrt{2})$ is a normal biquadratic extension of Galois group of type $(2, 2)$ and sub-quadratic extensions $K, K_0$ and $L = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$, by [16], we show that

$$h_2(F'_3) = \frac{1}{2} q(F'_3/\mathbb{Q}(\sqrt{2})) h_2(K) h_2(K_0) h_2(L),$$

and since $h_2(K') = 4$, $q(F'_3/\mathbb{Q}(\sqrt{2})) = 1$ and $h_2(L) = 1$ (see [24]), then $h_2(F'_3) = 2 h_2(K_0)$, which gives $\# G_2 = 4 h_2(K_0)$.

**Corollary 30.** Let $K = \mathbb{Q}(\sqrt{-pq(2 + \sqrt{2})})$ where $p$ and $q$ are primes such that $p \equiv -q \equiv 5$ mod 8 and $(\frac{2}{q}) = -1$, then in each of the extensions $F'_i$, for $i \in \{1, 2, 3\}$, there are exactly two classes of $C_{K,2}$ which capitulated and $G_2 \simeq Q_4$. Moreover, the 2-class group of $F'_3$ is cyclic of order 8, and the 2-class groups of $F'_2$ and $F'_1$ are of type $(2, 2)$.

**Proof.** Since $p \equiv -q \equiv 5$ mod 8, then, by Theorem 27, $G_2$ is quaternionic of order $2^m$ with $m > 3$. Let $K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{-pq})$, since $(\frac{2}{q}) = -1$, then, according to [17], we have $h_2(K_0) = 4$, by the remark 29, we find that $G_2 \simeq Q_4$ and the 2-class group of $F'_3$ is of order 8. So the 2-class group of $F'_3$ is cyclic of order 8, and the 2-class groups of $F'_2$ and $F'_1$ are of type $(2, 2)$. 

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Example 31. Let \( K = \mathbb{Q}(\sqrt{-15(2 + \sqrt{2})}) \), \( F'_1 = K(\sqrt{5}) \), \( F'_2 = K(\sqrt{-3}) \) and \( F'_3 = K(\sqrt{-15}) \), then by corollary 30 in each extension \( F'_i \), \( i \in \{1, 2, 3\} \), there are exactly two classes of \( C_{K,2} \) who capitulated. Moreover, in this case, the 2-class group of \( F'_3 \) is cyclic of order 8 and the 2-class groups of \( F'_1 \) and \( F'_2 \) are of type \((2, 2)\) and \( G_2 \) is quaternionic of order 16.

References


